Lecture 4 on Sept. 16

In the lecture today, we assume $f(z) = u(x, y) + iv(x, y)$ where u and v are real-valued functions. We call f is derivable at $z = z_0$ if the following limit exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Moreover we denote by $f'(z_0)$ the above limit. Notice here that in the 2-D complex plane, we have infinitely many ways to approach z_0. The above limit require a unique limit which is independent of how we approach z_0. Now we consider two specific limiting procedures. The first one is to approach z_0 along the x-axis, then we can assume $z = x_0 + h + iy_0$ where h is a real number. Therefore it holds

$$f'(z_0) = \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \, \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} = \partial_x u(z_0) + i \, \partial_x v(z_0).$$

We can also approach z_0 along the y-axis, then $z = z_0 + ih$. Furthermore it follows that

$$f'(z_0) = \lim_{h \to 0} \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{ih} + i \, \frac{v(x_0, y_0 + h) - v(x_0, y_0)}{ih} = -i \, \partial_y u(z_0) + \partial_y v(z_0).$$

By the above two equalities, if f is derivable at $z = z_0$, then

$$\partial_x u(x_0, y_0) = \partial_y v(x_0, y_0), \qquad \partial_x v(x_0, y_0) = -\partial_y u(x_0, y_0). \tag{0.1}$$

This is the so-called Cauchy-Riemann equation. In fact the Cauchy-Riemann equation also provides us with a criterion to determine if a complex function is derivable at z_0. That is

Theorem 0.1. *f is derivable at $z = z_0$ if and only if (0.1) holds at (x_0, y_0).*

With the above arguments, we have

Definition 0.2. *Given Ω a sub-domain in the complex plane, we call f is derivable or holomorphic on Ω if f is derivable at all points in Ω.*

Example 1: Find out all holomorphic functions on $\mathbb{C}$ whose imaginary part is 0;

Example 2: Find out all holomorphic functions on $\mathbb{C}$ whose real part is given by $u(x, y) = x^2 - y^2$;

Example 3: Find out all holomorphic functions on $\mathbb{C}$ whose real part is given by $u(x, y) = e^x \cos y$.

Example 4: Show that $\sin(xy)$ can not be the real part of some holomorphic functions on $\mathbb{C}$.

We have another criterion to determine if a function is derivable. Given $z = x + iy$, we have

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i}. \tag{0.2}$$

Then by the chain rule, formally we have

$$\partial_{\bar{z}} = \partial_x \frac{\partial x}{\partial \bar{z}} + \partial_y \frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\partial_x - \frac{1}{2i}\partial_y.$$

Therefore for a given function f, it holds

$$\partial_{\bar{z}} f = \frac{1}{2}\partial_x(u + iv) - \frac{1}{2i}\partial_y(u + iv) = \frac{1}{2}\left(\partial_x u - \partial_y v\right) + \frac{i}{2}\left(\partial_x v + \partial_y u\right).$$

Applying the Cauchy-Riemann equation (0.1) to the above equality, we know that if f is derivable in some domain Ω, then in Ω we have $\partial_{\bar z} f \equiv 0$. In other words, after the change of variables in (0.2), holomorphic functions admit only variable z in the expression of f.

Example 5: Now we consider Example 2 above. In this example, we know that $f = x^2 - y^2 + 2xyi$ is holomorphic on $\mathbb{C}$. By (0.2), we know that

$$f = \left(\frac{z+\bar z}{2}\right)^2 - \left(\frac{z-\bar z}{2i}\right)^2 + 2\left(\frac{z+\bar z}{2}\right)\left(\frac{z-\bar z}{2i}\right) i = z^2.$$

Clearly we know that f depends only on variable z.

Example 6: $f = |z|^2$ is not holomorhpic since $f = z\bar z$ which tells us that $\partial_{\bar z} f = z \neq 0$. $f = x$ is not derivable neither in that $f = (z+\bar z)/2$ which shows that $\partial_{\bar z} f = 1/2 \neq 0$.

Remark 0.3. *From Theorem 0.1 and Definition 0.2, we know that the differentiability of a complex function is quite different from the differentiability of real-valued functions. For $f(z) = u(x,y) + i\,v(x,y)$, u and v must satisfy the Cauchy-Riemann equation so that the defined function f can be derivable. Usually even though u and v are smooth real-valued functions, but the complex function $f = u + i\,v$ can be non-derivable (c.f. Example 6). The Cauchy-Riemann equation is a very restrictive condition for a function to be derivable. However on the other hand, we might have more elegant features for the family of holomorphic functions.*

One of the most important features satisfied by holomorphic functions are the following maximum modulus theorem. we just state the result without proof.

Theorem 0.4. *If f is holomorphic on Ω, then $|f(z)|$ achieve its maximum value on $\partial\Omega$.*

Now we begin to study some elementary functions. Firstly let us consider polynomials.

Definition 0.5. *We call $P_n(z) = a_0 + a_1 z + \dots + a_n z^n$ a polynomial of order n if $a_n \neq 0$.*

Two properties of polynomials are important to us. The first one is

Theorem 0.6 (Fundamental Theorem of Algebra)**.** *Suppose that $P_n(z)$ is a polynomial of order n. Then $P_n(z) = 0$ has at least one solution in $\mathbb{C}$.*

By induction, we can show from Theorem 0.6 that

Corollary 0.7. *Any polynomial of order n can be factorized as follows:*

$$P_n(z) = c\,\Pi_{k=1}^{n}(z - \alpha_k).$$

Here we give a proof of Theorem 0.6 by Theorem 0.4.

Proof of Theorem 0.6 . If $P_n(z)$ has no solution in $\mathbb{C}$, then

$$Q(z) = \frac{1}{P_n(z)}$$

is well defined and holomorphic on $\mathbb{C}$. By Theorem 0.4, we have

$$\sup_{B_R(0)} |Q(z)| \leq \sup_{|x|=R} |Q(z)| \longrightarrow 0, \qquad \text{as } R \longrightarrow \infty.$$

Here $B_R(0)$ is the disk centering at 0 with radius R. The above limit shows that $Q(z) \equiv 0$. Or equivalently $P_n(z) \equiv \infty$. But this is impossible. The proof is finished. $\square$